POST-CRITICAL BEHAVIOR OF A SIMPLE NON-LINEAR SYSTEM IN A CROSS-WIND

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Many theoretical models of slender prismatic beams in a cross-wind have been developed during last decades. They mostly follow various types of the linear approach. Therefore their applicability is very limited especially for prediction of the system post-critical behavior. The subject considered in this paper represents a part of a complex theoretical background of the general nonlinear model which would enable to predict any system reaction in the pre- and post-critical domain. In particular, the aeroelastic self-induced oscillation of a mechanical system with generalized single degree of freedom (SDOF) is discussed. The motion is described by an ordinary differential equation of Duffing type with special generalized aero-elastic damping of Van der Pol type. A new semi-analytical approach is introduced to identify the limit cycles both stable and unstable. The latter are not possible to be identified by means of experiments nor by the numerical integration.

Keywords: *limit cycles, dynamic stability, post-critical effects, non-linear dynamics*

1. Introduction

The vibration and the stability of the prismatic body in an air flow is often a result of the aero-elastic interaction of the response and the forces varying in time which have the non-conservative and gyroscopic nature. Several types of aero-elastic oscillation are known using technical language in the wind engineering: flutter, galloping or self-excited motion induced by separating vortices at the body or in the wake of it. Each of them can be observed separately depending on geometry and mechanical properties of the structure and the flow conditions. Very often however, the distinction may not be so clear. A bluff body structure can display both flutter and galloping characteristics (which may be also affected by the wind turbulence). For example, during the life of the Tacoma Narrows bridge the experience with the vertical and torsional vibration had been associated with the galloping at first. Later, torsional flutter combined with wake vortices was identified to cause the collapse, see e.g. [1]. General view at the conditions of dynamic stability and physical interpretation is given in [2], where several types of aero-elastic stability loss known from engineering practice and experimental data are identified in the domain created by the eigen-frequencies of individual response components.

During the past decades, many mathematical explanations of the aero-elastic phenomenon have been suggested. However all of them suffer from uncertainty in determination of the the role of individual parameters being fundamental for the instability origin in a particular technical branch. Moreover, the analysis has revealed relatively considerable diversity of conclusions following from experimental studies. This is probably due to historical treat-

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ment of these problems in a number of branches as well as due to the existence of various instability domains and bifurcation points types. In the course of time, however, the research has succeeded in understanding that the majority of the models have either obvious or hidden linear character, because they are based on various types of convolution formulations. Some of these approaches are able to predict lower limits of aero-elastic stability, but they do not provide any possibility to investigate the strongly non-linear post-critical behavior. In any case, the detailed knowledge of the post-critical state is very important, because it may explain a possible secondary re-stabilization due to non-linear effects. They are able at least for a limited period of time, to restitute one of lower types of stability after the structure has lost its exponential or asymptotic stability, see [3]. In [4] the model of the parametric excitation of the aero-elastic oscillators is considered in a form of Mathieu nonlinear equation. The possibility of re-stabilization of the trivial solution after up-crossing the critical state is shown.

There are many papers dealing with single degree-of-freedom aero-elastic system. The seminal work is considered to be the paper [5]. This paper deals with the quasi-steady analysis of transverse galloping of a long square prism in a normal steady wind. The aerodynamic force coefficient is modeled as the polynomial of the seventh degree and the respective equation is solved by Krylov and Bogolyubov method. More recently, other paper appeared. In [6] for example, galloping oscillation with a single rotational degree-of-freedom is investigated with the unsteady aerodynamic forces taken as a combination of aerodynamic stiffness and damping terms. In the paper [7], the aerodynamic yaw moment is written in a form of Taylor-Fourier expansion and subsequently a mathematical model similar to the Van der Pol equation is created. The stable and unstable limit cycles are discussed.

The paper [8] describes the torsional flutter mechanism of 2D rectangular cylinders and 2D H-shaped cylinders based upon unsteady pressure measurements under forced torsional vibration. For cylinders with fixed side ratio in the high reduced velocity range, the torsional flutter mechanism is in principle identical with that of coupled flutter. However, in the low reduced velocity range the torsional flutter is induced by vortices at along side-surface of the cylinder and thus differs from coupled flutter mechanism. In the paper [9], the square cross-section beam in a normal steady flow is analyzed with regard to the global stability conditions. There are several methods used and compared with the numerical integration in order to predict bifurcation points. The stability is treated also in [10]. Authors evaluated the branches of periodic solutions and their stability as functions of wind velocity. The existence of quasi-periodic solutions is proved.

The subject considered in this paper is the non-linear dynamics of a mechanical oscillator with one degree of freedom in a wind field arising from reduction of the model described

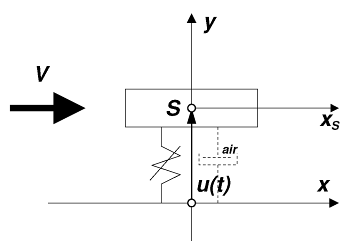

Fig.1: Model of the self-excited aero-elastic SDOF structure with nonlinear stiffness and aero-elastic damping due to non-conservative and gyroscopic forces

in [11]. The schematic picture of the system can be viewed at the Figure 1. It has been shown that such system has several types of bifurcation manifolds. The analysis of existence and relevant portraits of the principal limit cycles have been carried out. In this paper, the new approach is introduced. The gyroscopic forces are written in a form of the polynomial of the second and consequently of the fourth order assume the possibility of occurring also a non-stable limit cycles, which are not possible to be identified by means of experiments or numerical integration. The general bifurcation equation is developed by means of analytical formula.

2. Problem outline

The relevant equation for depicted SDOF system can be written in a general form:

$$\ddot{u} + g(u) = \mu\, f(u, \dot{u}) \ . \tag{1}$$

The right hand side of the equation, the aerodynamic forcing function $f(u, \dot{u})$ depends on the geometry and the wind speed. Generalizing the common harmonic assumption for the response, we assume the solution in a form

$$u(t) = a \cos\varphi(t) + b \ . \tag{2}$$

In this formula, the generalized phase $\varphi(t)$ is used. It can be naturally explained as a function given by a relation $\mathrm{d}\varphi(t)/\mathrm{d}t = \Phi(\varphi)$. The generalized frequency $\Phi(\varphi)$ is the periodic function with the period 2π. The coefficient a is the amplitude of the function and the coefficient b determines the eccentricity of the response with respect to the origin. Using the chain differentiation rule, we may rewrite Eq. (1) in the following way:

$$\Phi\, \frac{\mathrm{d}}{\mathrm{d}\varphi}(\Phi\, u') + g(u) = \mu\, f(u, \Phi\, u') \ , \qquad \text{where:} \qquad u' = \frac{\mathrm{d}u}{\mathrm{d}\varphi} \ . \tag{3}$$

Eq. (3) represents the transform of Eq. (1) into the coordinate $\varphi(t)$. The both sides of the Eq. (3) should be multiplied by $u' = -a \sin\varphi$ and integrated across the interval $\widetilde{\varphi} \in \langle 0, \varphi\rangle$. This leads to the formula:

$$\int_0^{\varphi} \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\widetilde{\varphi}}(\Phi\, u')^2\right) \mathrm{d}\widetilde{\varphi} + \int_0^{\varphi} g(u)\, \mathrm{d}u = -a\, \mu \int_0^{\varphi} f(u, \Phi u')\, \sin\widetilde{\varphi}\, \mathrm{d}\widetilde{\varphi} \ . \tag{4}$$

Hence the fundamental expression, which enables to deduce the resulting equation for the establishment of the limit cycles is obtained:

$$\frac{1}{2}\,(\Phi\, a\, \sin\varphi)^2 + h(a\, \cos\varphi + b) - h(a+b) = -\mu\, a \int_0^{\varphi} f(a\, \cos\widetilde{\varphi} + b, -a\, \Phi\, \sin\widetilde{\varphi})\, \sin\widetilde{\varphi}\, \mathrm{d}\widetilde{\varphi} \ . \tag{5}$$

The function h in this formula represents the indefinite integral of $g(u)$ i.e. $h(u) = \int g(u)\, \mathrm{d}u$. Evaluating Eq. (5) for: (i) $\widetilde{\varphi} \in \langle 0, 2\pi\rangle$ and (ii) $\widetilde{\varphi} \in \langle 0, \pi\rangle$, two important conditions can be deduced respectively:

$$\left.\begin{aligned} \int_0^{2\pi} f(a_k, b_k, \varphi, \Phi_k)\, \sin\varphi\, \mathrm{d}\varphi = 0\ , \qquad &\text{(i)} \\ h(a_k + b_k) - h(-a_k + b_k) - a_k\,\mu \int_0^{\pi} f(a_k, b_k, \varphi, \Phi_{k-1})\, \sin\varphi\, \mathrm{d}\varphi = 0\ . \qquad &\text{(ii)} \end{aligned}\right\} \tag{6}$$

Eqs (6) allow to calculate values of a and b and approximation of $\Phi(\varphi)$ and hence the solution of non-linear equation in an iterative loop. This approximation is converging rapidly to the exact solution (calculated for example numerically). The balance should be evaluated after each half of the cycle due to possibility that $b \neq 0$, because the characteristics are not symmetric with respect to the origin. The period of the limit cycle is given by the formula:

$$T = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\Phi(\varphi)}\ . \tag{7}$$

Often, the first, or rather zero-*th*, approximation of the solution is important especially when it can be also calculated analytically. In this case, if the parameter $\mu \approx 0$, the first approximation of $\Phi_k = \Phi_0$, corresponding to the solution of homogeneous form of the Eq. (1) can be obtained, i.e.:

$$\Phi_0 = \left[\frac{2\,h(a_0 + b_0) - 2\,h(a_0\, \cos\varphi + b_0)}{a_0^2\, \sin^2\varphi}\right]^{\frac{1}{2}}\ . \tag{8}$$

Employing the right hand side with nonlinear damping, it can be deduced easily, that for the *k-th* approximation with $\mu > 0$ one may obtain the higher approximations of Φ as follows:

$$\Phi_k(\varphi) = \left[\frac{2\,h(a_k + b_k) - 2\,h(a_k\, \cos\varphi + b_k) - 2\,a_k\,\mu \displaystyle\int_0^{\varphi} f(a_k, b_k, \widetilde{\varphi}, \Phi_{k-1})\, \sin\widetilde{\varphi}\, \mathrm{d}\widetilde{\varphi}}{a_k^2\, \sin^2\varphi}\right]^{\frac{1}{2}}\ . \tag{9}$$

Knowing the generalized frequency, we are able to establish the stability of the periodic solution. According to the Floquet theory, it can be determined by an exponent λ, called multiplier of the periodic orbit which is characterising the phase volume evolution. It is given as the trace of Jacobi matrix of the system (1), i.e. by the formula:

$$\lambda = \int_0^{T} \frac{\partial f(u, \dot{u})}{\partial \dot{u}}\, \mathrm{d}t\ . \tag{10}$$

The negative value of λ implicates the stable limit cycle, while λ positive indicates that the limit cycle becamed unstable. If trajectory is stable according to Lyapunov, then arbitrary initial perturbation doesn't prove any grow, on average, along the trajectory.

3. Example-stable limit cycle

Let us demonstrate this approach on the equation describing the motion of the system with one degree-of-freedom (heave or rotation) oscillating in the flow, possibly reaching limit cycles oscillation. Firstly, we write the right hand side in the form of Van der Pol damping :

$$\ddot{u} + \alpha\, u + \beta\, u^3 = \mu\, (\eta - \nu\, u^2)\, \dot{u}\ , \tag{11}$$

where $\alpha,\ \beta,\ \nu,\ \eta$ are the equation coefficients which can have different values. In aeroelastic practice they should be ascertained experimentally in the wind tunnel. Applying Eq. (8) we obtain the first approximation of the generalized frequency $\Phi(\varphi)$ in an analytical form. It holds with the respect to Eq. (8) :

$$\Phi_0 = \left[\alpha + \frac{1}{2}\,\beta\, a_0^2\, (1 + \cos^2\varphi)\right]^{\frac{1}{2}}\ . \tag{12}$$

Integrating this formula, the generalized phase as well as the flow field in the plane $(u, \dot{u})$ can be calculated. Finally, the coefficient λ of the first approximation may be calculated by means of Eq. (10) as :

$$\lambda = \nu\,\sqrt{\frac{2}{\beta}}\int\limits_0^{2\pi}\frac{\varrho - a_0^2\,\cos^2\varphi}{(\delta + a_0^2\,(1+\cos^2\varphi))^{\frac{1}{2}}}\,\mathrm{d}\varphi\ ; \qquad \varrho = \frac{\eta}{\nu}\ ; \qquad \delta = \frac{2\,\alpha}{\beta}\ . \tag{13}$$

The λ coefficient depends strongly on the combination of the parameters ϱ and δ. As said before, it should be less than zero in order to observe the stable limit cycle.

4. Example-unstable limit cycle

By extension of the right hand side of the Eq. (11) we may demonstrate the behavior of the Eq. (1), now with the possibility of the existence of unstable limit cycle. The following adoption is carried out and discussed :

$$\ddot{u} + \alpha\, u + \beta\, u^3 = \mu\, (\eta - \nu\, u^2 + \vartheta\, u^4)\, \dot{u}\ . \tag{14}$$

It should be particularly noticed, that the right hand side of Eq. (14) includes the fourth degree of the response in order to encompass possibly both stable and unstable limit cycles as it has been observed also experimentally in a wind channel, see [11]. Their existence is predetermined by a particular ratio of parameters $\eta,\ \nu,\ \vartheta$. The theoretical solution of the above equation demonstrates the considerable sensitivity of the system to self-excited vibration with respect to particular values of parameters. The structure of the right-hand side of the governing equation leads to the oscillation around the origin $(0, 0)$, thus the value of b vanishes. However, including the terms with odd powers of u on the right hand side of Eq. (14), the shift of the origin in the phase plane and consequently the nonzero value of b will appear.

The advantage of the proposed method emerges, when both stable (the cases when the bifurcation parameter $\vartheta < 0$) as well as unstable ($\vartheta_{\max} > \vartheta > 0$) limit cycles should be investigated. This method is able to depict both types of limit cycles (if they exist) as it is shown at Figure 2. The stability diagram complementary to the graphs of limit cycles is given on the same figure, where the bifurcation diagram for certain η and ν is shown.

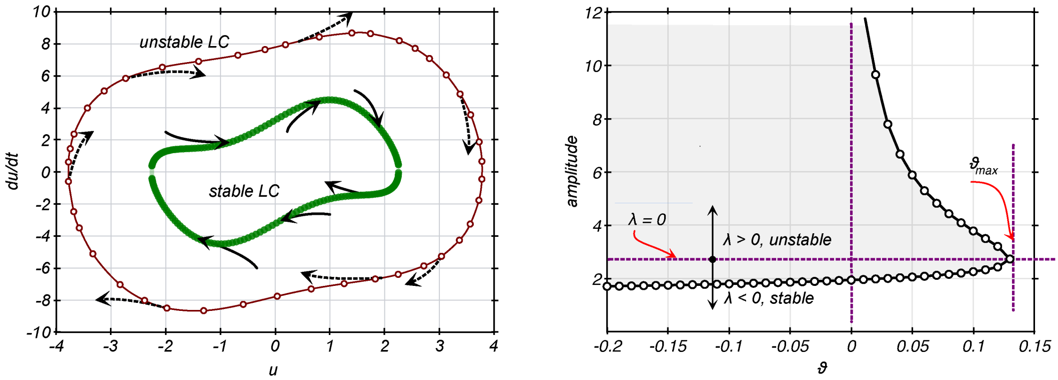

Fig.2: Attractive and repulsive limit cycles as a solution of Eq. (14); left: stable and unstable limit cycles meets at the certain point creating the separatrix manifold; right: stability diagram $a = f(\vartheta)$ for ϑ varying in the interval $\vartheta \in \langle -0.2, \vartheta_{\max} \rangle$

The detailed knowledge of both types of limit cycles is very important. The stable cycle acts as an attractor. Whatever initial conditions putting the system inwards the stable cycle lead to the trajectory with an increasing amplitude asymptotically approaching the cycle from inside. Similarly the trajectory approaches asymptotically the stable cycle if initial conditions lie in the strip between the stable and unstable cycles. When an energy supply results in a system state abandoning the instable limit cycle, the response amplitude increases exponentially beyond all limits and the system collapses. Therefore, the unstable limit cycle represents a limit separating convergent and divergent cases. On the other hand a short excess doesn't have lead to collapse. The parameter λ is decisive, see Figure 2, right part. If the integral Eq. (13) remains negative although the trajectory indicates some excesses beyond the outer limit cycle, the system response converges to the stable cycle. It is coming to light, that three intervals of ϑ should be distinguished, if $\eta > 0$, $\nu > 0$: (i) $\vartheta < 0$ – one stable cycle exists and any arbitrary initial conditions lead to stable response being represented by this one; (ii) $0 \leq \vartheta < \vartheta_{\max}$ – two limit cycles arise as discussed; (iii) $\vartheta > \vartheta_{\max}$ – no stable solution exists whatever are the initial conditions.

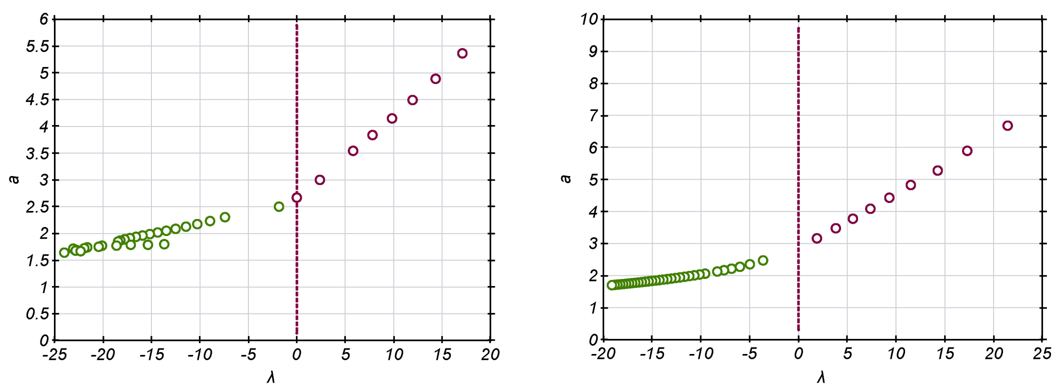

Fig.3: The amplitude a as a function of λ; red circles are the amplitudes of unstable LC, whereas green circles represent the stable ones; left: $\alpha = 0.1$, $\beta = 0.1$, $\nu = 1$, $\eta = 1$; right: $\alpha = -0.1$, $\beta = 0.1$, $\nu = 1$, $\eta = 1$; vertical lines at both graphs ($\lambda = 0$) represents the point at the separatrix

Let us look at the case when the stiffness terms, the function $g(u) = \alpha\, u + \beta\, u^3$, vary. Two cases are shown. First, it has only one zero point at the point $u = 0$, i.e. $\alpha > 0$, $\beta > 0$. Such a case is shown at Figure 3-left part. The second case is illustrated at Figure 3-right part, with the function crossing the axis u not only at the origin. This means that $\alpha < 0$ and $\beta > 0$.

5. Bifurcation equation and diagrams

The role of the zero-*th* approximation is very important and cannot be neglected. Moreover, it can be often obtained analyticaly. It can be seen in the following explanation, analyzing equation :

$$\ddot{u} + \alpha\, u = \mu\,(\eta - \nu\, u^2 + \vartheta\, u^4)\,\dot{u}\ . \tag{15}$$

This equation is similar to Eq. (14) omitting Duffing member. However, it reveals that Duffing member plays a minor role in the stability character of limit cycles. We use again the solution $u(t) = a\, \cos\varphi$. This time, however, φ and a are simple variables and not functions of time. Multiplying Eq. (15) by $u'(t)$ and integrating both sides, we obtain the equation :

$$0 = a\,\mu \int\limits_0^{2\pi} \left(a^4\,\vartheta\, \cos^4(\varphi) - a^2\,\nu\, \cos^2(\varphi) + 1\right) \sin^2(\varphi)\, \mathrm{d}\varphi\ . \tag{16}$$

Now it is easy to deduce, that this condition results in the relationship we name the governing bifurcation formula for this particular problem. Evaluating integral Eq. (16) it holds :

$$1 - \frac{1}{4}\,\nu\, a^2 + \frac{1}{8}\,\vartheta\, a^4 = 0\ . \tag{17}$$

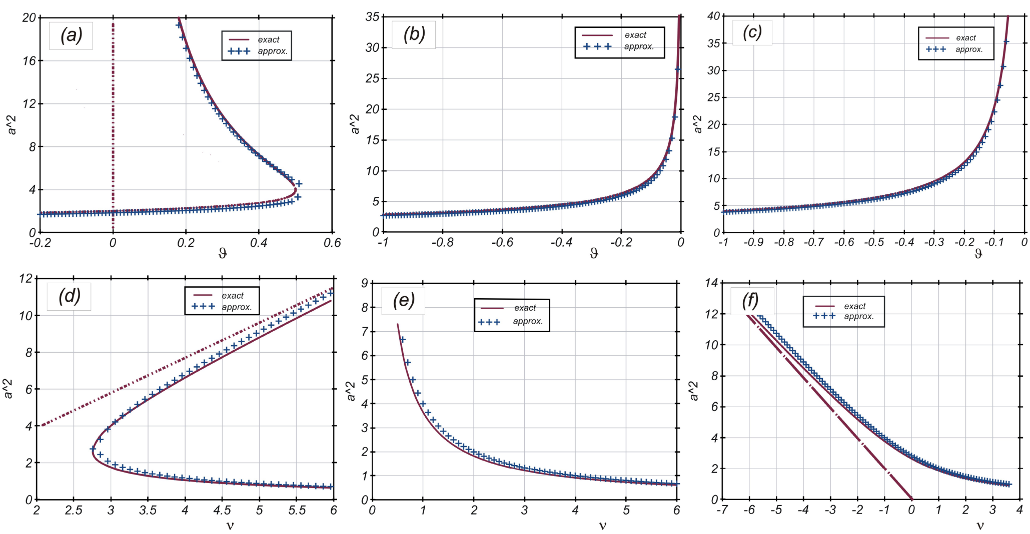

Fig.4: The plot of bifurcation equation; the squared of amplitude a is a function of parameters ϑ or ν respectively, while the other key parameter is kept constant; the upper row : $a^2 = f(\vartheta)$ for a) $\nu > 0$, b) $\nu = 0$ and c) $\nu < 0$; the second row is presenting the plot of $a^2 = f(\nu)$, when d) $\vartheta > 0$, e) $\vartheta = 0$ and f) $\vartheta < 0$

The diagrams at the Figure 4 show several plots of bifurcation equation. The dash-dotted (blue) curves represent the solution given by the zero-th approximation, while the full (red) curves represent the exact solutions corresponding to the final shape of limit cycles.

The upper part of Figure 4 shows, that positive ϑ is admissible only if ν is positive as well. Otherwise ϑ has to be negative. Similar conclusions are obvious also from the lower part of Figure 4. Using these bifurcation diagrams aero-elastic stability of the system (14) or (15) can be easily assessed taking into account that all parameters α, β, η, ν, ϑ are functions of the stiffness and mainly of the cross section geometry and wind velocity.

6. Conclusions

After the loss of trivial solution stability the response in one degree of freedom tends to stabilize itself in the form of a stable limit cycle, see Figure 2-left. The solid (green) curve represents an attractor for all configurations of initial conditions, when parameter $\vartheta < 0$. For $\vartheta > 0$ stable ($\lambda < 0$) and unstable ($\lambda > 0$) limit cycles exist until the point ($\vartheta = \vartheta_{\max}$, $\lambda = 0$) is reached. This point provides a twofold limit cycle and represents a strong energy barrier. When this energy barrier is overcame by means of further wind energy supply, for example, the limit cycles don't exist any more and the final stability loss occurs. The system response starts to grow beyond all limits. This process can be initiated from the viewpoint of our analysis when a is increasing and inducing a transition of λ through the zero value. The twofold limit cycle making an important separatrix manifold cannot be directly determined neither by means of experiments nor the numerical integration. The approximative analytical formula can be used for the qualitative analysis of bifurcation diagrams. This means, that the shape of the limit cycle is disregarded in such a case. It can substitute the full numerical solution obtained by an iterative process.

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References

[1] Billah Y.K., Scanlan R.H.: Resonance, Tacoma Narrows bridge failure, and undergraduate physics textbooks, American Journal of Physics, 59 (2), 1991

[2] Náprstek J.: Domains and types of aeroelastic instability of slender beam, In: Procedings of the ASME IDETC 2007 - 21st Biennial Conf. Mech. Vibration and Noise, (H.S. Tzou and H. Jalili eds.), ASME, Las Vegas, CD paper #DETC2007-34132, 2007, 10 pgs.

[3] Tondl A.: Non-linearly coupled systems, In: Proceeding of the Conference Engineering Mechanics '99 (C. Kratochvíl et al. eds.), Svratka (CZ), 1999, pp. 87–96

[4] Lumbantobing, Haaker T.I.: On the parametric excitation of some nonlinear aeroelastic oscillators, Journal of Fluids and Structures, 19, 2004, pp. 221–237

[5] Parkinson G.V., Smith J.D.: The square prism as an aeroelastic non-linear oscillator, Quart. Journal of Mechanics and Applied Mathematics, Vol. XVII, 1963, pp. 225–239

[6] Oudheusden B.W.: Aerodynamic stiffness and damping effects in the rotational galloping of a rectangular cross-section, Journal of Fluids and Structures, 14, 2000, pp. 1119–1144

[7] Leech C.M.: Limit cycle stability of aerodynamically induced yaw oscillations, International Journal Mechanical Science, 1979, pp. 517–525

[8] Matsumoto M., Daito Y., Yoshizumi F., Ichikawa Y., Yabutani T.: Torsional flutter of bluff bodies, Journal of Wind Engineering and Industrial Aerodynamics, 69–71, 1997, pp. 871–882

[9] Vio G.A. Dimitriadis G., Cooper J.E.: Bifurcation analysis and limit cycle oscillation amplitude prediction methods applied to te aeroelastic galloping problem, Journal of Fluids and Structures, 23, 2007, pp. 983–1011
[10] Luongo A., Zulli D., Piccardo G.: Analytical and numerical approaches to nonlinear galloping of internally resonant suspended cables, Journal of Sound and Vibration, 315, 2008, pp. 373–393
[11] Náprstek J., Pospíšil S., Fischer C.: Post-critical limit cycles and nonstationary response types of aeroelastic system, In: Proc. International Conference on Flow-Induced Vibrations 2008 (I. Zolotarev, J. Horáček eds.), IT ASCR Prague, Prague, 2008, pp. 41–47

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